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The justification of approximate bounded solutions of mixed Sturm–Liouville problems

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Abstract

The approximate solution of a homogeneous singular integral equation, or equivalently a homogeneous infinite algebraic system, is fully justified and the error resulting from the truncation is estimated. This equation (algebraic system) is the standard form to which many even mixed Sturm–Liouville problems are reducible.

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1. Introduction

In the integral equation formulation of mixed Sturm–Liouville problems, proposed by Eckhardt and ElSheikh [1], the problem is reduced to a discrete Riemann problem, which in turn is transformed to the form of a class of homogeneous singular integral equations with Cauchy's kernel. The solution of this integral equation designates the unknown function in terms of its Fourier components that are still to be determined. To this end, the solution can be further reduced to a homogeneous infinite system of algebraic equations and as such it, and consequently the integral equation itself, can only be approximately solved. The approximation consists namely in truncating the homogeneous system (equation) and excluding all the components of the unknown function that exceed a certain order.

The justification of the above-mentioned truncation was expected through semi-analytic reasonings made in [1]. A numerical experiment was performed in [2] which confirmed these predictions. The first attempt to justify this approximation was started in [3]. It stands on the fact that a solution of a homogeneous system (equation) can be determined only to within a multiplicative factor. Thus, provided an eigenvalue is known, the corresponding solution can be so normalized that its first Fourier component is equal to unity and the homogeneous system (equation) in this way is equivalent to an inhomogeneous one (equation). This equivalence remains valid on performing the truncation at any order. At this point it is worthwhile to summarize the theorem of Chersky [4] used in that attempt to clarify why it was appropriate to suggest the present work, which stands again on this theorem.

Theorem 1. *Let the following conditions be fulfilled.*

(1) The approximate equation $\tilde{k}\tilde{\varphi} = \tilde{f}$ *has the unique solution* $\tilde{\varphi}$.
(2) $f - \tilde{f} \in Y_0$, *where* Y_0 *is a linear subset,* $Y_0 \subset Y$.
(3) Operator $K - \tilde{K}$ *is acting from* X *into* Y_0.
(4) The inverse operator $\tilde{K}^{-1}$, *acting from* Y_0 *into* $X_0 \subset X$, *is determined.*
(5) $\|\tilde{K}^{-1}(K - \tilde{K})\| < 1$.

Then the equation $K\varphi = f$, has the unique solution

$$\varphi = \tilde{\varphi} + [I + \tilde{K}^{-1}(K - \tilde{K})]^{-1}\tilde{K}^{-1}(f - K\tilde{\varphi})$$

and the following estimate holds:

$$\|\varphi - \tilde{\varphi}\|_{X_0} \leqslant \frac{\left\|\tilde{K}^{-1}(f - K\tilde{\varphi})\right\|_{X_0}}{1 - \|\tilde{K}^{-1}(K - \tilde{K})\|}.$$

The operator K considered in [3] was the integral operator of the inhomogeneous equation to which the Sturm–Liouville problem was reduced. In contrast to $K - \tilde{K}$, the operator $\tilde{K}^{-1}$ could not have been definitely estimated and thus the establishment of the justification is somewhat incomplete. Moreover, the error taking place in the solution of the integral equation due to the truncation could not have been estimated.

In this work the justification of the truncation applied to that homogeneous integral equation is completely established and a definite expression for the resulting error is given.

In section 2, the class of singular integral equations we are concerned with as well as the procedures of its solution are briefly exhibited. This solution is in turn viewed as another integral equation to which the above-mentioned theorem of Chersky can be applied. In section 3 are outlined steps of determining the estimations of which use has been made in the previous section. In section 4, numerical verifications are given. The typical problem used here is that considered in [3]: the Dirichlet–Neumann mixed Sturm–Liouville problem for the Laplacian in the unit disc. Section 5 is devoted to some conclusions and comments.

2. The class of homogeneous singular integral equations, its solution through truncation, justification and error estimation

In many initial Dirichlet–Neumann problems, the major task consists in solving the corresponding mixed Sturm–Liouville problem. Denoting by $\varphi_-(\theta;\gamma)$ the even Hölder-continuous extension of the Dirichlet condition corresponding to the eigenvalue γ and which is compatible with the Neumann condition where it is imposed, this extension is the solution of a form of the Cauchy-type integral equation [1–3]

$$\frac{1}{\pi\mathrm{i}}\int_{-c}^{c}\frac{\varphi_-(t;\gamma)}{1 - \mathrm{e}^{\mathrm{i}(\theta-t)}}\,\mathrm{d}t - Q_0(\gamma)\Phi_{0-}(\gamma)\theta - {\sum_{n=-\infty}^{\infty}}{}' \frac{1}{n}Q_n(\gamma)\Phi_{n-}(\gamma)\mathrm{e}^{\mathrm{i}n\theta} = \alpha \tag{2.1}$$

where α is a constant that can be set equal zero since it does not contribute to the solution and

$$Q_{|n|}(\gamma) = \mathrm{o}(|n|^{-1}). \tag{2.2}$$

In equation (2.1), the prime over the summation symbols indicates that the value $n = 0$ is not included and the principal interval $[-c, c]$ is the domain of the support of $\varphi_-(\theta;\gamma)$ while this function vanishes on $[-\pi, \pi]/[-c, c]$. It can also be shown that the above assertion holds true for problems with configurations other than these considered in [1–3] or even for problems with more complicated boundary conditions such as occur in elastic contact problems.

If the series in equation (2.1) converges to an L_2-function, then this equation has the following bounded and Hölder-continuous solution [3], [5, p 257]:

$$\varphi_-(\theta;\gamma) = -R(\theta)\Bigg\{Q_0(\gamma)\Phi_{0-}(\gamma)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}[I_n(\theta) - I_{-n}(\theta)] - \sum_{n=1}^{\infty}Q_n(\gamma)\frac{\Phi_{n-}(\gamma)}{n}[I_n(\theta) - I_{-n}(\theta)]\Bigg\} \tag{2.3}$$

where

$$R(\theta) = \lim_{\substack{z\to e^{i\theta}\\ |z|<1}}\sqrt{(z - e^{ic})(z - e^{-ic})} \tag{2.4}$$

$$I_n(\theta) = \frac{1}{\pi}\int_{-c}^{c}\frac{e^{i(n+1)t}}{R(t)(e^{it} - e^{i\theta})}\,dt \tag{2.5}$$

and the explicit expressions of the integrals $I_n(\theta)$ that are equivalent to those given in [1–3] are obtained from

$$-e^{-i\theta}I_{-n}(-\theta) = I_n(\theta) = -e^{i(n-1)\theta}\sum_{j=0}^{n-1}e^{-i\theta j}P_j(\cos c) \tag{2.6}$$

where $P_j(\cos c)$ are Legendre polynomials defined by the formula

$$P_j(\cos c) = \frac{1}{\pi}\int_{-c}^{c}\frac{e^{-ij\theta}}{R(\theta)}\,d\theta. \tag{2.7}$$

Further, it will turn out in the following section that

$$Y_n^2 = \|g_n(\theta)\|_{L_2}^2 = \|R(\theta)[I_n(\theta) - I_{-n}(\theta)]\|_{L_2}^2 = o\left(\frac{1}{n}\right) \tag{2.8}$$

and if

$$g(\theta) = \frac{R(\theta)}{\pi}\int_{-c}^{c}\frac{te^{it}dt}{R(t)(e^{it} - e^{i\theta})} = R(\theta)\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}[I_n(\theta) - I_{-n}(\theta)] \tag{2.9}$$

then

$$Y^2 = \|g(\theta)\|_{L_2}^2 < \infty. \tag{2.10}$$

To define the Fourier coefficients $\Phi_{n^-}(\gamma)$ and complete the definition of the solution (2.3), the Fourier transform can be applied to it and the following homogeneous algebraic system is obtained:

$$\begin{aligned}&\Phi_{0-}(\gamma) + N_0Q_0(\gamma)\Phi_{0-}(\gamma) - \sum_{n=1}^{\infty}\frac{Q_n(\gamma)}{n}[N_{n0} - N_{-n0}]\Phi_{n-}(\gamma) = 0\\ &\Phi_{\ell-}(\gamma) - \frac{1}{\ell}N_{\ell o}Q_0(\gamma)\Phi_{0-}(\gamma) - \sum_{n=1}^{\infty}\frac{Q_n(\gamma)}{n}[N_{n\ell} - N_{-n\ell}]\Phi_{n-}(\gamma) = 0\\ &\ell \in \mathsf{N}\end{aligned} \tag{2.11}$$

where the expressions of $N_{\pm n\ell}$ (for $n \in \mathsf{N}$ and $\ell \in \mathsf{N}^+$) and N_0 are given in [3] and can be extracted from the contents of the next section on deducing the important results

$$|N_{\pm n0}| < o\left(\frac{1}{n}\right) \qquad |N_{\pm n\ell}| < o\left(\frac{1}{n^{1/2}\ell^{3/2}}\right) \qquad \text{and} \qquad N_0 < \infty \quad (n, \ell \in \mathsf{N}). \tag{2.12}$$

It is clear that system (2.11) can be only approximately solved by means of the truncation. The small zeros of the truncated determinant represent good approximations for the small eigenvalues of the problem and the higher the order of the truncation the larger the upper bound of the precisely obtained eigenvalues. No new eigenvalue smaller than that upper bound appears however the truncation order is further increased while the corresponding extensions φ_- turn rapidly to be practically stable. Additionally, the approximate solutions of a certain problem, obtained in this way, were again used as a basis for redefining the eigenvalues by means of the Rayleigh–Ritz technique [6]. The improved eigenvalues are practically stable starting from the fourth-order truncated solutions and the improvements are almost negligible when the truncation exceeds the seventh order.

It should be noted that relations (2.12) together with (2.2) show that it is the first bounded number of terms from the summation in equations (2.11) that contribute to the value of $\Phi_{\ell-}(\gamma)$ and

$$\Phi_{\ell^-}(\gamma) < \mathrm{o}(\ell^{-3/2}) \tag{2.13}$$

in accordance with the requirement that $\tilde{\Phi}_{\ell-}(\gamma)$ are Fourier coefficients of a Hölder-continuous function. Thus, the summation in equation (2.1) is an $L_2[-c, c]$ function and this ensures the existence of its bounded solution (2.3).

If a certain eigenvalue γ is known, system (2.11) can in general be solved by setting one of its unknowns equal to unity ($\tilde{\Phi}_{0-}(\gamma)$, say) and excluding one of its equations (the first which corresponds to $\ell = 0$, say). The inhomogeneous and non-singular system thus obtained is exactly what could have been obtained on following the same technique starting from the inhomogeneous integral equation obtained by replacing $\tilde{\Phi}_{0-}(\gamma)$ by unity in equation (2.1) and the truncation of the latter yields the same algebraic system truncated at the same order [3, 6]. Thus, equation (2.1) can be rewritten as

$$\frac{1}{\pi}\int_{-c}^{c}\frac{\varphi_-(t;\gamma)}{1-\mathrm{e}^{\mathrm{i}(\theta-t)}}\,\mathrm{d}t - {\sum_{n=-\infty}^{\infty}}' Q_{|n|}(\gamma)\frac{\Phi_{n-}(\gamma)}{n}\mathrm{e}^{\mathrm{i}n\theta} = Q_0(\gamma)\theta \tag{2.14}$$

while the solution (2.3) assumes the integral equation form

$$\begin{aligned} K\varphi_-(\gamma) &= \varphi_-(\theta;\gamma) - \frac{R(\theta)}{2\pi^2}\int_{-c}^{c}\frac{\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t}{R(t)(\mathrm{e}^{\mathrm{i}t}-\mathrm{e}^{\mathrm{i}\theta})}\int_{-c}^{c}W(t-x)\varphi_-(x;\gamma)\,\mathrm{d}x \\ &= Q_0(\gamma)g(\theta) \end{aligned} \tag{2.15}$$

where

$$W(t) = {\sum_{n=-\infty}^{\infty}}' \frac{Q_n(\gamma)}{n}\mathrm{e}^{\mathrm{i}nt}. \tag{2.16}$$

The truncated operator can be defined by

$$\tilde{K}\varphi_-(\gamma) = \tilde{\varphi}_-(\theta;\gamma) - \frac{R(\theta)}{2\pi^2}\int_{-c}^{c}\frac{\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t}{R(t)(\mathrm{e}^{\mathrm{i}t}-\mathrm{e}^{\mathrm{i}\theta})}\int_{-c}^{c}{\sum_{n=-N}^{N}}'\frac{Q_{|n|}(\gamma)}{n}\mathrm{e}^{\mathrm{i}n(t-x)}\tilde{\varphi}_-(x;\gamma)\,\mathrm{d}x. \tag{2.17}$$

Thus, we can take $X = X_0 = Y = Y_0 = L_2[-c, c]$.

Lemma 1. *For any $\varepsilon > 0$, $\|K - \tilde{K}\| < \varepsilon$ provided N is appropriately chosen. Moreover*

$$\begin{aligned} \|K-\tilde{K}\| &\leqslant 2\sum_{n=N+1}^{\infty}\frac{Q_n(\gamma)}{n}\sqrt{\pi\left[2N_{n0}^2 + \sum_{\ell=1}^{\infty}(N_{n\ell}-N_{-n\ell})^2\right]}\frac{|\Phi_{n-}(\gamma)|}{\|\varphi_-(\gamma)\|} = U(N) \\ &\leqslant \sum_{n=N+1}^{\infty}\frac{\mathrm{const}}{n^4}. \end{aligned} \tag{2.18}$$

Proof. For any φ_- in X_0 we have

$$\begin{aligned}
\|(K-\tilde{K})\varphi_-(\gamma)\|_{L_2} &= \left(\int_{-c}^{c}\left|\frac{R(\theta)}{2\pi^2}\int_{-c}^{c}\frac{\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t}{R(t)(\mathrm{e}^{\mathrm{i}t}-\mathrm{e}^{\mathrm{i}\theta})}\right.\right.\\
&\quad\left.\left.\times\int_{-c}^{c}\sum_{|n|>N}\frac{Q_n(\gamma)}{n}\mathrm{e}^{\mathrm{i}n(t-x)}\varphi_-(x;\gamma)\,\mathrm{d}x\right|^2\mathrm{d}\theta\right)^{1/2}\\
&= \left(\int_{-c}^{c}\left|R(\theta)\sum_{n=N+1}^{\infty}\frac{Q_n(\gamma)}{n}\Phi_{n-}(\gamma)[I_n(\theta)-I_{-n}(\theta)]\right|^2\mathrm{d}\theta\right)^{1/2}\\
&\leqslant \sum_{n=N+1}^{\infty}\frac{Q_n(\gamma)}{n}|\Phi_{n-}(\gamma)|\|R(\theta)[I_n(\theta)-I_{-n}(\theta)]\|\\
&\leqslant \left(\sum_{n=N+1}^{\infty}\frac{Q_n(\gamma)}{n}\|R(\theta)[I_n(\theta)-I_{-n}(\theta)]\|_{L_2}\frac{|\Phi_{n-}(\gamma)|}{\|\varphi_-(\gamma)\|}\right)\|\varphi_-(\gamma)\|\\
&\leqslant \left[2\sum_{n=N+1}^{\infty}\frac{Q_n(\gamma)}{n^{3/2}}\frac{|\Phi_{n-}(\gamma)|}{\|\varphi_-(\gamma)\|}\right]\|\varphi_-(\gamma)\|
\end{aligned}$$

where the last step is obtained in view of the estimate (2.8). The required has been proved. □

The solution of the equation

$$\tilde{K}\tilde{\varphi}_- = Q_0(\gamma)g(x) \tag{2.19}$$

is reached on determining the coefficients $\tilde{\Phi}_{n-}(\gamma)$, $n = 1, 2, \ldots, N$, and is achieved by applying the complex finite Fourier transform to it. This yields a similar system to (2.11) but truncated at order N and in which

$$\tilde{\Phi}_{0-}(\gamma) = 1. \tag{2.20}$$

The first equation of the system thus obtained will be automatically fulfilled by the solution of the rest of the equations together with (2.20) as N approaches infinity [3, 6]. This latter solution exists in general since γ is a zero of the determinant of the whole system (2.11), which differs from that of the rest of the system even when N increases indefinitely. In this way we finally have

$$\begin{aligned}
\tilde{\varphi}_-(\theta;\gamma) = -R(\theta)\Bigg(&Q_0(\gamma)\sum_{n=1}^{\infty}\frac{(-)^n}{n}[(I_n(\theta)-I_{-n}(\theta)]\\
&-\sum_{n=1}^{N}Q_n(\gamma)\frac{\tilde{\Phi}_{n-}(\gamma)}{n}[(I_n(\theta)-I_{-n}(\theta)]\Bigg).
\end{aligned} \tag{2.21}$$

Lemma 2. *The inverse operator $\tilde{K}^{-1}$ is bounded. Moreover*

$$\begin{aligned}
\left\|\tilde{K}^{-1}\right\| &\leqslant 1+\frac{1}{|Q_0(\gamma)|}\sum_{n=1}^{N}\left|\frac{Q_n(\gamma)}{n}\right|\alpha_n\left|\tilde{\Phi}_{n-}(\gamma)\right| = V(N)\\
&\leqslant 1+\sum_{n=1}^{N}\frac{\text{const}}{n^3}Q_n(\gamma)
\end{aligned} \tag{2.22}$$

where

$$\alpha_n = \frac{\|R(\theta)[I_n(\theta)-I_{-n}(\theta)]\|_{L_2}}{\|g(\theta)\|_{L_2}}. \tag{2.23}$$

Proof. We consider the norm

$$\|\tilde{\varphi}_-\|_{L_2} = |Q_0(\gamma)| \cdot \|\tilde{K}^{-1}g\|_{L_2}$$

$$= \left(\int_{-c}^{c} \left|\sum_{n=1}^{N} \frac{Q_n(\gamma)}{n}\tilde{\Phi}_{n-}(\gamma)R(\theta)[I_n(\theta) - I_{-n}(\theta)] + Q_0(\gamma)g(\theta)\right|^2 \mathrm{d}\theta\right)^{1/2}$$

$$\leqslant \sum_{n=1}^{N}\left(\int_{-c}^{c}\left|\frac{Q_n(\gamma)}{n}\tilde{\Phi}_{n-}(\gamma)R(\theta)[I_n(\theta) - I_{-n}(\theta)]\right|^2 \mathrm{d}\theta\right)^{1/2}$$

$$+|Q_0(\gamma)|\|g(\theta)\|_{L_2}$$

$$\leqslant \sum_{n=1}^{N}\left|\frac{Q_n(\gamma)}{n}\right||\Phi_{n-}(\gamma)|\left(\int_{-c}^{c}|R(\theta)[I_n(\theta) - I_{-n}(\theta)]|^2\right)^{1/2}$$

$$+|Q_0(\gamma)|\|g(\theta)\|_{L_2}$$

$$\leqslant \sum_{n=1}^{N}\left|\frac{Q_n(\gamma)}{n}\right||\Phi_{n-}(\gamma)|\alpha_n\|g(\theta)\|_{L_2} + |Q_0(\gamma)|\|g(\theta)\|_{L_2}.$$

The lemma has been proved. □

It is now clear that the inequality

$$\|K - \tilde{K}\|\|\tilde{K}^{-1}\| < 1$$

holds true under appropriate choice of N and we finally have the following theorem.

Theorem 2. *Under the condition $U(N)V(N) < 1$, equation (2.1) has a unique solution subject to condition (2.20) in $L_2[-c, c]$ corresponding to every eigenvalue γ.*

The function $\tilde{\varphi}_-(\theta;\gamma)$ defined by formula (2.21) is an approximate solution of equation (2.1) and the following estimation holds:

$$\|\varphi_-(\gamma) - \tilde{\varphi}_-(\gamma)\|_{L_2} \leqslant \frac{U(N)V(N)}{1 - U(N)V(N)}\|\tilde{\varphi}_-(\gamma)\|_{L_2}. \tag{2.24}$$

3. The proof of the estimates

The estimates we have used in the previous section stand in turn on two basic estimates: the first [7, formula (22.14.9)]

$$|P_n(\cos c)| \leqslant \left(\frac{2}{\pi \sin c}\right)^{1/2}\frac{1}{\sqrt{n}} \qquad n \in N \quad P_0(\cos c) = 1 \tag{3.1}$$

and the second [1]

$$|A_{-1}| = |A_{-2}| = \left|\frac{\cos c}{2} - \frac{1}{2}\right| \leqslant 1$$

and

$$|A_{-k}| = \left|\frac{1}{2^k}\left[\frac{(2k-5)!![\mathrm{e}^{-\mathrm{i}(k-1)c} + \mathrm{e}^{\mathrm{i}(k-1)c}]}{(k-1)!} - \sum_{m=1}^{k-2}\frac{(2m-3)!!(2k-2m-5)!!}{m!(k-1-m)\mathrm{e}^{\mathrm{i}(k-1-2m)c}}\right]\right|$$

$$\leqslant \frac{2(2k-5)!!}{(2k-2)!!} \qquad k > 2 \quad [(-1)!! = 0!! = 1]. \tag{3.2}$$

Thus

$$|N_{-1\ell}| = |A_{-\ell-2}||P_0(\cos c)| \leqslant \frac{2(2\ell-1)!!}{(2\ell+2)!!} < \frac{1}{(\ell+1)\sqrt{\ell}}$$

$$|N_{-2\ell}| = |A_{-\ell-3}||P_0(\cos c)| + \left(\frac{2}{\pi \sin c}\right)^{1/2} |A_{-\ell-2}|$$
$$\leqslant \frac{1}{(\ell+2)\sqrt{\ell+1}} + \left(\frac{2}{\pi \sin c}\right)^{1/2} \frac{1}{(\ell+1)\sqrt{\ell}}$$

and for $n \in \mathsf{N} - \{1, 2\}$ and $\ell \in \mathsf{N}$ we have

$$|N_{-n\ell}| \leqslant \left|A_{-n-\ell-1}\right|\left|P_0(\cos c)\right| + \sum_{j=1}^{n-1} \left|A_{j-n-\ell-1}\right|\left|P_j(\cos c)\right|$$
$$\leqslant \frac{1}{(n+\ell)(n+\ell-1)^{1/2}} + \left(\frac{2}{\pi \sin c}\right)^{1/2} \sum_{j=1}^{n-1} \frac{1}{(n+\ell-1-j)^{3/2}\sqrt{j}}$$
$$\leqslant \frac{1}{(n+\ell)(n+\ell-1)^{1/2}} + 2\left(\frac{2}{\pi \sin c}\right)^{1/2} \int_1^{\sqrt{n-1}} \frac{\mathrm{d}x}{(n+\ell-1-x^2)^{3/2}}$$
$$= \frac{1}{(n+1)(n+\ell-1)^{1/2}}$$
$$+2\left(\frac{2}{\pi \sin c}\right)^{1/2} \frac{1}{(n+\ell-1)}\left[\frac{\sqrt{n-1}}{\sqrt{\ell}} - \frac{1}{\sqrt{n+\ell-2}}\right] \tag{3.3}$$

and analogous results for $N_{n\ell}$. For $\ell = 0$ we obtain in a similar way

$$|N_{10}| = |N_{-1o}| = |A_{-2}| < 1$$

$$|N_{20}| = |N_{-20}| \leqslant |A_{-3}| + \left(\frac{2}{\pi \sin c}\right)^{1/2} |A_{-2}| \leqslant \frac{1}{4} + \left(\frac{2}{\pi \sin c}\right)^{1/2}$$

$$|N_{30}| = |N_{-30}| \leqslant |A_{-4}| + \left(\frac{2}{\pi \sin c}\right)^{1/2} \left(|A_{-3}| + \frac{1}{\sqrt{2}}|A_{-2}|\right)$$
$$\leqslant \frac{1}{8} + \left(\frac{2}{\pi \sin c}\right)^{1/2} \left(\frac{1}{4} + \frac{1}{\sqrt{2}}\right)$$

$$|N_{n0}| = |N_{-n0}| \leqslant |A_{-n-1}| + \sum_{j=1}^{n-2} \left|A_{j-n-1}\right|\left|P_j(\cos c)\right| + |A_{-2}||P_{n-1}(\cos c)|$$
$$\leqslant \frac{1}{n\sqrt{n-1}} + 2\left(\frac{2}{\pi \sin c}\right)^{1/2} \left(\frac{\sqrt{n-2}}{(n-1)} - \frac{1}{(n-1)\sqrt{n-2}} + \frac{1}{\sqrt{n-1}}\right) \tag{3.4}$$

and

$$|N_0| < 2\sum_{n=1}^{\infty} \frac{|N_{-n0}|}{n} < \infty. \tag{3.5}$$

Taking into account that $N_{n\ell}$, for example, is the ℓth Fourier component of the function $R(\theta)I_n(\theta)$ we can use the Parseval equality to obtain

$$Y_n^2 = \|R(\theta)[I_n(\theta) - I_{-n}(\theta)]\|_{L_2}^2 = 4\pi\left[2|N_{n0}|^2 + \sum_{\ell=1}^{\infty} (N_{n\ell} - N_{-n\ell})^2\right] \tag{3.6}$$

and equation (2.8) follows. In the same way, since N_0 and $\frac{1}{n}N_{-n0}$ are the Fourier coefficients of the function $g(\theta)$ defined by equation (2.9) it follows that

$$Y^2 = \|g(\theta)\|_{L_2}^2 = 2\pi N_0^2 + 4\pi \sum_{n=1}^{\infty} \frac{N_{-n0}^2}{n^2} \tag{3.7}$$

which establishes the result (2.10).

4. Illustrative numerical example

In this section the above results are applied to the Sturm–Liouville problem considered in [3] for which

$$Q_n(\gamma) = \frac{\gamma J_{n+1}(\gamma)}{J_n(\gamma)}.$$

In the case $c = \frac{\pi}{2}$ the first three eigenvalues of this problem are

$$\gamma_1 = 1.2445 \qquad \gamma_2 = 2.9437 \qquad \text{and} \qquad \gamma_3 = 4.6047.$$

The evolution of $|Q_n(\gamma_i)|$, with respect to n and γ, is exhibited in table 1.

As expected, these coefficients terminate with n as a whole although it increases with respect to γ since $Q_n(\gamma) = \frac{\gamma^2}{2(n+1)} + \cdots$. At the eigenvalues γ_i, the nominators of $Q_n(\gamma)$ may not vanish since its zeros are the eigenvalues of the limiting case $c = 0$, the uniform Dirichlet problem. Nevertheless, it may happen that some eigenvalue of our problem (for example γ_3) nears some zero(s) of these denominators. At such eigenvalues the values of some coefficients $Q_n(\gamma)$ grow and slow down the decay of the quantity $U(N)V(N)$ (see equations (2.18), (2.22) and (2.24)). However, $Q_n(\gamma)$ cannot assume large values if n exceeds a certain limit. We may recall that the larger the order of a Bessel function the larger the magnitude of its first zero. For our considered eigenvalues, $Q_n(\gamma)$ decrease monotonically starting from $n = 3$ and their values are exceeded by unity if $n \geqslant 10$. Thus, to ensure regular results for the analysis of section 2 we can take $N \geqslant 10$.

In table 2 values of Y and Y_n are calculated by means of expressions (3.7) and (3.5) respectively. In the summations included by both expressions, only the first 120 terms are considered. The same results to five decimals are obtained when decreasing the number of the terms considered to 100. The same hold true for the values

$$\|\varphi_-(\gamma_1)\| = 3.7183 \qquad \|\varphi_-(\gamma_2)\| = 5.2456 \qquad \text{and} \qquad \|\varphi_-(\gamma_3)\| = 3.6728$$

which are calculated from the Parseval's relation

$$\|\varphi_-(\gamma_i)\| \approx \sqrt{2\pi\left[1 + 2\sum_{n=1}^{120} \tilde{\Phi}_{n-}^2(\gamma_i)\right]}.$$

The values of $U_i(N)(= U(N, \gamma_i))$ as well as $V_i(N)$ and $E_i(N)$ obtained in the same way are given in tables 3, 4 and 5 respectively, where

$$E(N) = \frac{U(N)V(N)}{1 - U(N)V(N)}.$$

Table 1. The evolution of $Q_n(\gamma_i)$.

i	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=10$	$n=20$	$n=60$	$n=90$	$n=120$
1	0.997	0.414	0.266	0.197	0.156	0.064	0.035	0.012	0.008	0.006
2	−4.405	30.966	1.815	1.227	0.938	0.366	0.218	0.071	0.047	0.035
3	4.023	−3.269	10.48	3.977	2.669	0.915	0.487	0.173	0.116	0.087

Table 2. The value of $\|g(\theta)\|$ to the left and the behaviour of $\|g_n(\theta)\|$ as n increases.

Y	$Y_n=1$	$Y_n=2$	$Y_n=3$	$Y_n=10$	$Y_n=20$	$Y_n=60$	$Y_n=100$
2.553	3.779	2.828	2.983	2.608	2.566	2.530	2.518

Table 3. Upper estimations of the norm of the neglected (remainder) operator $K(\gamma_i) - \tilde{K}(N,\gamma_i)$, $i=1,2,3$ and $N=10,20,30$.

i	$N=10$	$N=20$	$N=30$
1	0.060 28	0.020 05	0.000 02
2	0.001 51	0.000 26	0.000 09
3	0.004 01	0.000 72	0.000 26

Table 4. Upper estimations of the norm of the inverse truncated operator $\tilde{K}^{-1}(N,\gamma_i)$, $i=1,2,3$ and $N=10,20,30$.

i	$N=10$	$N=20$	$N=30$
1	1.576	1.581	1.582
2	2.803	2.811	2.813
3	2.605	2.622	2.626

Table 5. The upper bounds $E(N,\gamma_i)$, $i=1,2,3$ and $N=10,20,30$ of the resulting error in the solution of the homogeneous integral equation due to the truncation.

i	$N=10$	$N=20$	$N=30$
1	0.000 46	0.000 08	0.000 03
2	0.004 26	0.000 76	0.000 27
3	0.010 55	0.001 91	0.000 69

5. Conclusion

The mixed Sturm–Liouville problems can be formulated into singular homogeneous integral equations. The eigenvalues of such equations can be exactly found but the corresponding solutions can in general be approximated. To make them useful, the error resulted in these solutions must be estimated and the approximation of a homogeneous operator must be definitely justified *a priori*. This represents a rather difficult requirement to satisfy. To the author's best knowledge, there are no references in the literature about the approximation of a homogeneous operator except possibly his previous incomplete attempt involved in [3]. In this work, the above-mentioned requirement is fulfilled. The lines established here can be followed when analogous results are required for different homogeneous operators.

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